KAM theory and Celestial Mechanics
5. KAM theorem: sketch of the proof and applications

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Outline

1. Sketch of the Proof for CS systems
2. The a-posteriori approach
3. Break–down of quasi–periodic tori and attractors
4. KAM break–down criterion
5. Applications

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Step 1: approximate solution and linearization
Step 2: determine the new approximation
Step 3: solve the cohomological equation
Step 4: convergence of the iterative step
Step 5: local uniqueness

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• Analytic tools:

- exponential decay of Fourier coefficients of analytic functions;
- estimates to bound the derivatives in smaller domains;
- quantitative analysis of the cohomology equations;
- abstract implicit function theorem.

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Step 1: approximate solution and linearization

• Let (K, μ) be an approximate solution: $f_\mu \circ K(\theta) - K(\theta + \omega) = E(\theta)$.

• Using the Lagrangian property in coordinates, $DK^T(\theta)\, J \circ K(\theta)\, DK(\theta) = 0$, the tangent space is

$$\text{Range}\Big(DK(\theta)\Big) \oplus \text{Range}\Big(V(\theta)\Big)$$

with $V(\theta) = J^{-1} \circ K(\theta)\, DK(\theta) N(\theta)$ and $N(\theta) = (DK(\theta)^T DK(\theta))^{-1}$.

• Define:

$$M(\theta) = [DK(\theta) \mid V(\theta)]\ .$$

Lemma

Up to a remainder R:

$$Df_\mu \circ K(\theta)\, M(\theta) = M(\theta+\omega) \begin{pmatrix} \mathrm{Id} & S(\theta) \\ 0 & \lambda \mathrm{Id} \end{pmatrix} + R(\theta)\,. \qquad (R)$$

Proof: Recall $M(\theta) = [DK(\theta) \mid V(\theta)]$.
Part 1: taking the derivative of $f_\mu \circ K(\theta) = K(\theta+\omega) + E(\theta)$, one gets $Df_\mu \circ K(\theta)\, DK(\theta) = DK(\theta+\omega) + DE(\theta)$;
Part 2: due to the remark on the tangent space, one has:

$$Df_\mu \circ K(\theta)\, V(\theta) = DK(\theta+\omega)\, S(\theta) + V(\theta+\omega)\, \lambda \mathrm{Id} + h.o.t.$$

with

$$\begin{aligned} S(\theta) \ &\equiv \ N(\theta+\omega)^T DK(\theta+\omega)^T Df_\mu \circ K(\theta)\, J^{-1} \circ K(\theta)\, DK(\theta) N(\theta) \\ &- \ N(\theta+\omega)^T DK(\theta)^T J^{-1} \circ K(\theta) DK(\theta) N(\theta+\omega) \lambda \mathrm{Id}\,. \end{aligned}$$

Step 2: determine a new approximation $K' = K + MW$, $\mu' = \mu + \sigma$ satisfying

$$f_{\mu'} \circ K'(\theta) - K'(\theta + \omega) = E'(\theta) . \qquad (APPR - INV)'$$

• Expanding in Taylor series:

$$f_\mu \circ K(\theta) + Df_\mu \circ K(\theta)\, M(\theta) W(\theta) + D_\mu f_\mu \circ K(\theta)\sigma$$
$$-K(\theta + \omega) - M(\theta + \omega)\, W(\theta + \omega) + h.o.t. = E'(\theta) .$$

• Recalling that $f_\mu \circ K(\theta) - K(\theta + \omega) = E(\theta)$, the new error E' is quadratically smaller provided:

$$Df_\mu \circ K(\theta)\, M(\theta) W(\theta) - M(\theta + \omega)\, W(\theta + \omega) + D_\mu f_\mu \circ K(\theta)\sigma = -E(\theta) .$$

• Combine the previous formula

$$Df_\mu \circ K(\theta)\, M(\theta) W(\theta) - M(\theta+\omega)\, W(\theta+\omega) + D_\mu f_\mu \circ K(\theta)\sigma = -E(\theta)$$

and the Lemma:

$$Df_\mu \circ K(\theta)\, M(\theta) = M(\theta+\omega) \begin{pmatrix} \mathrm{Id} & S(\theta) \\ 0 & \lambda \mathrm{Id} \end{pmatrix} + R(\theta)\,, \qquad (R)$$

to get equations for $W = (W_1, W_2)$ and σ:

$$\underline{M(\theta+\omega)} \begin{pmatrix} \mathrm{Id} & S(\theta) \\ 0 & \lambda \mathrm{Id} \end{pmatrix} W(\theta) - \underline{M(\theta+\omega)}\, W(\theta+\omega) = -E(\theta) - D_\mu f_\mu \circ K(\theta)\sigma\,.$$

• Multiplying by $M(\theta+\omega)^{-1}$ and writing $W=(W_1,W_2)$, one gets

$$\begin{pmatrix} \mathrm{Id} & S(\theta) \\ 0 & \lambda \mathrm{Id} \end{pmatrix} \begin{pmatrix} W_1(\theta) \\ W_2(\theta) \end{pmatrix} - \begin{pmatrix} W_1(\theta+\omega) \\ W_2(\theta+\omega) \end{pmatrix} = \begin{pmatrix} -\tilde E_1(\theta) - \tilde A_1(\theta)\sigma \\ -\tilde E_2(\theta) - \tilde A_2(\theta)\sigma \end{pmatrix} .$$

with $\tilde E_j(\theta) = -(M(\theta+\omega)^{-1}E)_j$, $\tilde A_j(\theta) = (M(\theta+\omega)^{-1}D_\mu f_\mu \circ K)_j$.

• In components:

$$\begin{aligned} W_1(\theta) - W_1(\theta+\omega) &= -\widetilde E_1(\theta) - S(\theta)W_2(\theta) - \widetilde A_1(\theta)\,\sigma \qquad (A) \\ \lambda W_2(\theta) - W_2(\theta+\omega) &= -\widetilde E_2(\theta) - \widetilde A_2(\theta)\,\sigma \qquad (B) \end{aligned}$$

• Cohomological eq.s with constant coefficients for (W_1, W_2), σ for known S, $\widetilde{E} \equiv (\widetilde{E}_1, \widetilde{E}_2)$, $\widetilde{A} \equiv [\widetilde{A}_1 | \widetilde{A}_2]$:

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• (A) involves small (zero) divisors, since for $k=0$ one has $1 - e^{ik\cdot\omega} = 0$ in

$$W_1(\theta) - W_1(\theta+\omega) = \sum_k \widehat{W}_{1,k}\, e^{ik\cdot\theta} (1 - e^{ik\cdot\omega}) \ .$$

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• Non–degeneracy condition: computing the averages of eqs. (A), (B), determine $\langle W_2 \rangle$, σ by solving $(W_2 = \langle W_2 \rangle + B^0 + \sigma \tilde{B}^0)$

$$\begin{pmatrix} \langle S \rangle & \langle S B^0 \rangle + \langle \widetilde{A}_1 \rangle \\ (\lambda - 1)\mathrm{Id} & \langle \widetilde{A}_2 \rangle \end{pmatrix} \begin{pmatrix} \langle W_2 \rangle \\ \sigma \end{pmatrix} = \begin{pmatrix} -\langle S \tilde{B}^0 \rangle - \langle \widetilde{E}_1 \rangle \\ -\langle \widetilde{E_2} \rangle \end{pmatrix} \ .$$

Step 3: solve the cohomological equations

- Non–average parts of W_1, W_2: solve cohomological equations of the form

$$\lambda w(\theta) - w(\theta + \omega) = \eta(\theta)$$

with $\eta : \mathbb{T}^n \to \mathbb{C}$ known and with zero average.

Lemma

Let $|\lambda| \in [A, A^{-1}]$ for $0 < A < 1$, $\omega \in \mathcal{D}(C, \tau)$, $\eta \in \mathcal{A}_\rho$, $\rho > 0$ or $\eta \in H^m$, $m \geq \tau$, and

$$\int_{\mathbb{T}^n} \eta(\theta)\, d\theta = 0 \; .$$

Then, there is one and only one solution w with zero average and

$$\begin{aligned} \|w\|_{\mathcal{A}_{\rho-\delta}} &\leq C_6\, C\, \delta^{-\tau} \|\eta\|_{\mathcal{A}_\rho} \;, \\ \|w\|_{H^{m-\tau}} &\leq C_7\, C\, \|\eta\|_{H^m} \;. \end{aligned}$$

Sketch of the proof. Expand η as

$$\eta(\theta) = \sum_{j\in\mathbb{Z}^n} \widehat{\eta}_j e^{2\pi i j\cdot\theta}$$

and using

$$\lambda w(\theta) - w(\theta+\omega) = \eta(\theta)$$

find

$$\widehat{w}_j = (\lambda - e^{2\pi i j\cdot\omega})^{-1}\,\widehat{\eta}_j\ ;$$

when $\lambda = 1$, $j = 0$, it must be $\widehat{\eta}_0 = 0$.

Estimate the multipliers using Cauchy bounds and use the Diophantine condition ([Rüssmann]).

Step 4: convergence of the iterative step

• The invariance equation is satisfied with an error quadratically smaller, i.e.

$$\|E'\|_{\mathcal{A}_{\rho-\delta}} \leq C_8 \delta^{-2\tau} \|E\|^2_{\mathcal{A}_\rho} \ , \qquad \|E'\|_{H^{m-\tau}} \leq C_9 \|E\|^2_{H^m} \ .$$

• The procedure can be iterated to get a sequence of approximate solutions, say $\{K_j, \mu_j\}$. Convergence: through an *abstract implicit function theorem,* alternating the iteration with carefully chosen smoothings operators defined in a scale of Banach spaces (analytic functions or Sobolev spaces).

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Step 5: local uniqueness

• Under smallness conditions, if there exist two solutions (K_a, μ_a), (K_b, μ_b), then there exists $\psi \in \mathbb{R}^n$ such that

$$K_b(\theta) = K_a(\theta + \psi) \qquad \text{and} \qquad \mu_a = \mu_b \, .$$

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The a-posteriori approach

• Following [LGJV2005], for conformally symplectic systems, by adjusting the parameters under a suitable non-degeneracy condition *near an approximately invariant torus, there is a true invariant torus*, [CCL].

• A KAM theory with adjustment of parameters was developed in [Moser1967], but with a parameter count different than in [CCL], since [Moser1967] is very general and does not take into account the geometric structure.

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Advantages of the a-posteriori approach:

► it can be developed in any coordinate frame, not necessarily in action-angle variables;
► the system is not assumed to be nearly integrable;
► instead of constructing a sequence of coordinate transformations on shrinking domains as in the perturbation approach, we shall compute suitable corrections to the embedding and the drift.

Consequences of the a-posteriori approach for conformally symplectic systems (with R. Calleja, R. de la Llave):

- the method provides an efficient algorithm to determine the breakdown threshold, very suitable for computer implementations;
  - very refined quantitative estimates;
- local behavior near quasi–periodic solutions;
- partial justification of Greene's criterion (also with C. Falcolini);
- a bootstrap of regularity, which allows to state that all smooth enough tori are analytic, whenever the map is analytic;
- analyticity domains of the quasi–periodic attractors in the symplectic limit;
- whiskered tori for conformally symplectic systems.

Outline

Break–down of quasi–periodic tori and attractors

• We can compute a rigorous lower bound of the break–down threshold of invariant tori by means of KAM theory.

• Which is the real break–down value?

• In physical problems one can compare KAM result with a measure of the parameter. For example in the 3-body problem, $\varepsilon = \frac{m_{Jupiter}}{m_{Sun}} \simeq 10^{-3}$.

• In model problems one needs to apply numerical techniques: KAM break–down criterion, Greene's technique, frequency analysis, etc.

Outline

KAM break–down criterion [Calleja, Celletti 2010]

• Solve the invariance equation for (K, μ):

$$f_\mu \circ K(\theta) = K(\theta + \omega) \ .$$

• Numerically efficient criterion: close to breakdown, one has a blow up of the Sobolev norms of a trigonometric approximation of the embedding:

$$K^{(L)}(\theta) = \sum_{|\ell| \leq L} \widehat{K}_\ell \, e^{i\ell\theta} \ .$$

• A regular behavior of $\|K^{(L)}\|_m$ as ε increases (for λ fixed) provides evidence of the existence of the invariant attractor. Table: ε_{crit} for $\omega_r = 2\pi \frac{\sqrt{5}-1}{2}$.

Conservative case	Dissipative case	
ε_{crit}	λ	ε_{crit}
0.9716	0.9	0.9721
	0.5	0.9792

• Greene's method: breakdown of $\mathcal{C}(\omega)$ related to the stability of $\mathcal{P}(\frac{p_j}{q_j}) \to \mathcal{C}(\omega)$, but in the dissipative case: drift in an interval - *Arnold tongue* - admitting a periodic orbit.

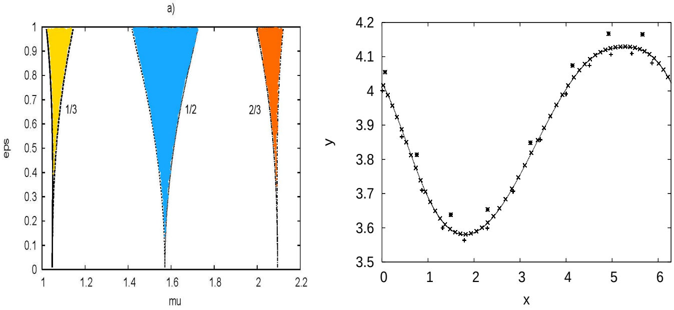

Figure: Left: Arnold's tongues providing μ vs. ε for 3 periodic orbits. Right: For $\lambda = 0.9$ and $\varepsilon = 0.5$ invariant attractor with frequency ω_r and approximating periodic orbits: 5/8 ($*$), 8/13 ($+$), 34/55 ($\times$).

• Greene's method: let $\varepsilon^{\omega_r}_{p_j,q_j}$ be the maximal ε for which the periodic orbit has a stability transition; the sequence converges to the breakdown threshold of $\omega_r = 2\pi\frac{\sqrt{5}-1}{2}$.

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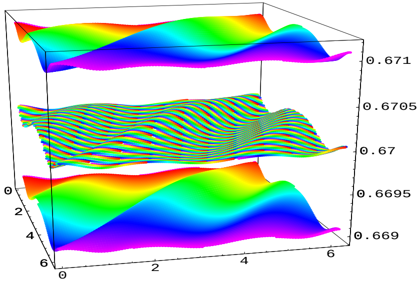
Outline

Applications

- Standard map
- Rotational dynamics: spin–orbit problem
- Orbital dynamics: three–body problem

KAM stability through confinement

• Confinement in 2–dimensional systems: dim(phase space)=4, dim(constant energy level)=3, dim(invariant tori)=2 → confinement in phase space for ∞ times between bounding invariant tori



• Confinement no more valid for $n > 2$: the motion can diffuse through invariant tori, reaching arbitrarily far regions (Arnold's diffusion).

Conservative standard map

Results of the '90s

• [A.C., L. Chierchia] Let $\omega = 2\pi\frac{\sqrt{5}-1}{2}$; $|\varepsilon| \leq 0.838$ (86% of Greene's value) there exists an invariant curve with frequency ω.
• [R. de la Llave, D. Rana] Using accurate strategies and efficient computer–assisted algorithms, the result was improved to 93% of Greene's value.
• Very recent results [J.-L. Figueras, A. Haro, A. Luque] in http://arxiv.org/abs/1601.00084 reaching 99.9%!!!

Dissipative standard map

- Using $K_2(\theta) = \theta + u(\theta)$, the invariance equation is

$$D_1 D_\lambda u(\theta) - \varepsilon \sin(\theta + u(\theta)) + \omega(1-\lambda) - \mu = 0 \qquad (1)$$

with $D_\lambda u(\theta) = u(\theta + \frac{\omega}{2}) - \lambda u(\theta - \frac{\omega}{2})$.

Proposition [dissipative standard map, R. Calleja, A.C., R. de la Llave (2016)]

Let $\omega = 2\pi \frac{\sqrt{5}-1}{2}$ and $\lambda = 0.9$; then, for $\varepsilon \le \varepsilon_{KAM}$, there exists a unique solution $u = u(\theta)$ of (1), provided that $\mu = \omega(1-\lambda) + \langle u_\theta \, D_1 D_\lambda u \rangle$.

- The drift μ must be suitably tuned and cannot be chosen independently from ω.

Dissipative standard map

- Using $K_2(\theta) = \theta + u(\theta)$, the invariance equation is

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• Preliminary result: conf. symplectic version, careful estimates, continuation method using the Fourier expansion of the initial approximate solution $\Rightarrow$

$$\varepsilon_{KAM} = \boxed{99\% \text{ of the critical breakdown threshold .}}$$

Rotational dynamics

The **Moon** and all evolved satellites, always point the same face to the host planet: 1:1 resonance, i.e. 1 rotation = 1 revolution (Phobos, Deimos - Mars, Io, Europa, Ganimede, Callisto - Jupiter, Titan, Rhea, Enceladus, etc.).
Only exception: **Mercury** in a 3:2 spin–orbit resonance (3 rotations = 2 revolutions).

• Important dissipative effect: **tidal torque,** due to the non–rigidity of planets and satellites.

Conservative spin–orbit problem

• Spin–orbit problem: triaxial satellite $\mathcal{S}$ (with $A < B < C$) moving on a Keplerian orbit around a central planet $\mathcal{P}$, assuming that the spin–axis is perpendicular to the orbit plane and coincides with the shortest physical axis.

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• Equation of motion:

$$\ddot{x} + \varepsilon\left(\frac{a}{r}\right)^3 \sin(2x - 2f) = 0\,, \qquad \varepsilon = \frac{3}{2}\frac{B-A}{C}\,.$$

• The (Diophantine) frequencies of the bounding tori are for example:

$$\omega_- \equiv 1 - \frac{1}{2 + \frac{\sqrt{5}-1}{2}}\,, \qquad \omega_+ \equiv 1 + \frac{1}{2 + \frac{\sqrt{5}-1}{2}}\,.$$

Proposition [spin–orbit model, A.C. (1990)]

Consider the spin–orbit Hamiltonian defined in $U \times \mathbb{T}^2$ with $U \subset \mathbb{R}$ open set. Then, for the true eccentricity of the Moon $e = 0.0549$, there exist invariant tori, bounding the motion of the Moon, for any $\varepsilon \leq \varepsilon_{Moon} = 3.45 \cdot 10^{-4}$.

Dissipative spin–orbit problem

• Possible forthcoming estimates: spin–orbit equation with tidal torque given by

$$\ddot{x} + \varepsilon \left(\frac{a}{r}\right)^3 \sin(2x - 2f) = -\lambda(\dot{x} - \mu) \ , \qquad (2)$$

where λ, μ depend on the orbital (e) and physical properties of the satellite.

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where λ, μ depend on the orbital (e) and physical properties of the satellite.

Proposition [A.C., L. Chierchia (2009)]

Let $\lambda_0 \in \mathbb{R}_+$, ω Diophantine. There exists $0 < \varepsilon_0 < 1$, such that for any $\varepsilon \in [0, \varepsilon_0]$ and any $\lambda \in [-\lambda_0, \lambda_0]$ there exists a unique function $u = u(\theta, t)$ with $\langle u \rangle = 0$, such that

$$x(t) = \omega\, t + u(\omega t, t)$$

solves the equation of motion with $\mu = \omega\, (1 + \langle u_\theta^2 \rangle)$.

Conservative three–body problem

• Consider the motion of a small body (with negligible mass) under the gravitational influence of two primaries, moving on Keplerian orbits about their common barycenter (*restricted* problem).

• Assume that the orbits of the primaries are circular and that all bodies move on the same plane: *planar, circular, restricted three–body problem* (PCR3BP).

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• Assume that the orbits of the primaries are circular and that all bodies move on the same plane: *planar, circular, restricted three–body problem* (PCR3BP).
• Adopting suitable normalized units and action–angle Delaunay variables $(L, G) \in \mathbb{R}^2$, $(\ell, g) \in \mathbb{T}^2$, we obtain a 2 d.o.f. Hamiltonian function:

$$\mathcal{H}(L, G, \ell, g) = -\frac{1}{2L^2} - G + \varepsilon R(L, G, \ell, g) \ .$$

• ε primaries' mass ratio ($\varepsilon = 0$ Keplerian motion). Actions: $L = \sqrt{a}$, $G = L\sqrt{1 - e^2}$.

• Degenerate Hamiltonian, but Arnold's isoenergetic non–degenerate (persistence of invariant tori on a fixed energy surface), i.e. setting $h(L, G) = -\frac{1}{2L^2} - G$:

$$\det \begin{pmatrix} h''(L,G) & h'(L,G) \\ h'(L,G)^T & 0 \end{pmatrix} = \det \begin{pmatrix} -\frac{3}{L^4} & 0 & \frac{1}{L^3} \\ 0 & 0 & -1 \\ \frac{1}{L^3} & -1 & 0 \end{pmatrix} = \frac{3}{L^4} \neq 0 \quad \text{for all } \ L \neq 0 \, .$$

• Dimension phase space = 4 , fix the energy: dim = 3; dimension invariant tori = 2.

Result: The stability of the small body can be obtained by proving the existence of invariant surfaces which confine the motion of the asteroid on a preassigned energy level.

Sample: Sun, Jupiter, asteroid 12 Victoria with

$$a_{\rm V} \simeq 0.449\ , \quad e_{\rm V} \simeq 0.220\ , \quad \imath_{\rm V} \simeq \frac{8.363 - 1.305}{360} = 1.961 \cdot 10^{-2}\ .$$

• Size of the perturbing parameter: $\varepsilon_J = 0.954 \cdot 10^{-3}$.
• Approximations: disregard $e_J = 4.82 \cdot 10^{-2}$ (worst physical approximation), inclinations, gravitational effects of other bodies (Mars and Saturn), dissipative phenomena (tides, solar winds, Yarkovsky effect,...)

• Concrete example: Sun, Jupiter, asteroid 12 Victoria with $a = 0.449$ (in Jupiter–Sun unit distance) and $e = 0.22$, so that $L_{\rm V} \simeq 0.670$, $G_{\rm V} \simeq 0.654$.

• Select the energy level as $E^*_{\rm V} = -\frac{1}{2L_{\rm V}^2} - G_{\rm V} + \varepsilon_J \langle R(L_{\rm V}, G_{\rm V}, \ell, g) \rangle \simeq -1.769$, where $\varepsilon_J \simeq 10^{-3}$ is the observed Jupiter–Sun mass–ratio. On such (3–dim) energy level prove the existence of two (2–dim) trapping tori with frequencies $\omega_\pm$.

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Proposition [three–body problem, A.C., L. Chierchia (2007)]

Let $E = E^*_{\rm V}$. Then, for $|\varepsilon| \leq 10^{-3}$ the unperturbed tori with trapping frequencies $\omega_\pm$ can be analytically continued into KAM tori for the perturbed system on the energy level $\mathcal{H}^{-1}\left(E^*_{\rm V}\right)$ keeping fixed the ratio of the frequencies.

• Due to the link between a, e and L, G, this result guarantees that a, e remain close to the unperturbed values within an interval of size of order ε.

Corollary: The values of the perturbed integrals $L(t)$ and $G(t)$ stay close forever to their initial values $L_{\rm V}$ and $G_{\rm V}$ and the actual motion (in the mathematical model) is nearly elliptical with osculating orbital values close to the observed ones.